

REPRESENTABILITY OF CHERN-WEIL FORMS

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ABSTRACT. In this paper we look at two naturally occurring situations where the following question arises. When one can find a metric so that a Chern-Weil form can be represented by a given form ? The first setting is semi-stable Hartshorne-ample vector bundles on complex surfaces where we provide evidence for a conjecture of Griffiths by producing metrics whose Chern forms are positive. The second scenario deals with a particular rank-2 bundle (related to the vortex equations) over a product of a Riemann surface and the sphere.

1. INTRODUCTION

In [18] the author introduced a geometric question : when one can find a metric so that a Chern-Weil form can be represented by a given form in the correct cohomology class ? Apart from a few very special cases (all of which involve conformally changing a given metric to another one) the answer is not known in general.

In this paper we study this question in the context of two natural situations in algebraic geometry and mathematical physics. The first of these is related to a conjecture of Griffiths dealing with Hartshorne-ample vector bundles. A holomorphic vector bundle E over a compact complex manifold X is said to be Hartshorne-ample if the canonical line bundle $\mathcal{O}_E(1)$ over $\mathbb{P}(E)$ is ample. If E admits a metric whose curvature Θ is Griffiths-positive, i.e., $\langle v, \Theta v \rangle$ is a positive $(1, 1)$ -form for all vectors $v \in E$, then it is easy to see that it is Hartshorne-ample. The converse is the Griffiths conjecture. In the case of curves this was proven in [20, 4]. Somewhat strong evidence for this conjecture in the general case is provided by the fact that if E is Hartshorne-ample, then $E \otimes \det(E)$ is Nakano-positive (stronger than Griffiths positive) [1, 16]. It is also known [2, 11] that the Schur polynomials of Hartshorne-ample bundles are numerically positive (and in fact the only numerically positive characteristic classes are positive linear combinations of the Schur polynomials [11]). In the case of bundles over surfaces, this means that $c_1^2, c_2, c_1, c_1^2 - c_2$ are numerically positive (and generate the “numerically positive cone”). It is but natural to ask whether there is a metric on E such that the corresponding Chern-Weil forms are positive. The following theorem addresses this question in the case of compact complex surfaces.

Theorem 1.1. *Let X be a compact complex surface and E be a Hartshorne-ample rank- r holomorphic vector bundle on it. Assume that E is semi-stable with respect*

to some polarisation $[L]$. Then there exists a hermitian metric h on E whose Chern-Weil forms satisfy $c_1(h) > 0, c_2(h) > 0, c_1^2(h) - c_2(h) > 0$.

Remark 1.1. It is worth noting that the assumption of semi-stability is actually quite natural in this context. The reason is that Umemura's proof [20] in the case of curves uses the concept of stability.

The second question we deal with arises from the gravitating vortex equations, which are themselves a special case of the Kähler-Yang-Mills equations studied in [5, 6, 7]. The Kähler-Yang-Mills equations for a metric H on a bundle E and a Kähler metric ω on a manifold X are

$$(1.1) \quad \begin{aligned} \sqrt{-1}\Theta \wedge \omega^{n-1} &= \lambda \omega^n \text{Id} \\ S_\omega - c &= \alpha \frac{\text{ch}_2(E, H) \wedge \omega^{n-2}}{\omega^n} \end{aligned}$$

where c is a constant, S_ω the scalar curvature of ω , Θ is the curvature of H , and $\text{ch}_2(E, H)$ is the second Chern character form. These equations admit a moment map interpretation just like the usual Hermite-Einstein equation and the constant scalar curvature Kähler (cscK) equation [5]. Special cases of these equations have been studied. In particular, in [6] a perturbation result (theorem 1.1) around $\alpha = 0$ was proven for $\Sigma \times \mathbb{P}^1$ (where Σ is a genus ≥ 1 Riemann surface) equipped with a certain $SU(2)$ -invariant rank 2-vector bundle (that we dub "the vortex bundle"). For $\mathbb{P}^1 \times \mathbb{P}^1$ an obstruction was found (akin to K-stability). Solving this equation in general is obviously quite challenging. However, taking cue from the Calabi volume conjecture which is easier than the problem of cscK metrics, intersects with cscK metrics in the Ricci flat case, and has no obstruction, we propose to study the following Calabi-Yang-Mills equations.

$$(1.2) \quad \begin{aligned} \sqrt{-1}\Theta \wedge \omega^{n-1} &= \lambda \omega^n \text{Id} \\ \omega^n - \eta &= \alpha \text{ch}_2(E, H) \wedge \omega^{n-2} \end{aligned}$$

where $\eta > 0$ is a given volume form in the right cohomology class. Note that if $\alpha = 0$ this is simply the usual Calabi conjecture. In this paper, the following existence result is proven for these equations.

Theorem 1.2. *Let Σ be a Riemann surface and $X = \Sigma \times \mathbb{P}^1$. Let $SU(2)$ act trivially on X and in the standard manner on $\mathbb{P}^1 = SU(2)/U(1)$. Let L be a holomorphic line bundle on Σ and E be a rank 2 holomorphic bundle over X which is an extension :*

$$(1.3) \quad 0 \rightarrow \pi_1^* L \rightarrow E \rightarrow \pi_2^* \mathcal{O}(2) \rightarrow 0.$$

Let $\tau > 0$ be a constant, $\omega_{FS} = \frac{idz \wedge d\bar{z}}{(1+|z|^2)^2}$ the Fubini-Study metric on \mathbb{P}^1 , and ω_Σ a metric on Σ . Denote by $\Omega = \pi_1^* \omega_\Sigma + \frac{1}{\tau} \pi_2^* \omega_{FS}$ an $SU(2)$ -invariant Kähler form on X where $\int_\Sigma \omega_\Sigma = \text{Vol}(\Sigma)$ is fixed, by H an $SU(2)$ -invariant hermitian metric on E , and by Θ the curvature of H . Also let α be a constant, $\text{ch}_2(E, H) = \frac{1}{2} \text{tr} \left(\frac{\sqrt{-1}\Theta}{2\pi} \right)^2$

be the second Chern character form, and η be an $SU(2)$ -invariant $(2, 2)$ -form on X . The following statements hold.

- (1) For any $\Omega > 0$ and $\eta > 0$ satisfying $\int_X \Omega^2 = \int_X \eta$, there exists an H such that $\Omega^2 + \alpha ch_2(E, H) = \eta$.
- (2) Assume that $0 < c_1(L) < \frac{\tau \text{Vol}(\Sigma)}{4\pi}$. Suppose we are given an $\eta > 0$ satisfying $\int_X \Omega^2 = \int_X \eta$. Then the set of $\alpha \geq 0$ satisfying
- $$(1.4) \quad 8 + \frac{2\alpha\tau}{(2\pi)^2} \left[2\lambda - \frac{\tau}{2} \right] > 0$$

for which there exists a smooth form $\Omega_\alpha > 0$ and a smooth metric H_α such that the Calabi-Yang-Mills equations are satisfied, i.e.,

$$(1.5) \quad \begin{aligned} \sqrt{-1}\Theta_\alpha \wedge \Omega_\alpha &= \lambda \Omega_\alpha^2 Id \\ \Omega_\alpha^2 + \alpha ch_2(E, H_\alpha) &= \eta, \end{aligned}$$

contains $\alpha = 0$ and is open. Moreover, for $\alpha = 0$ the solution is essentially unique among all $SU(2)$ -invariant solutions.

Notice that unlike the case of the Kähler-Yang-Mills equations, there seems to be no difference between the case of $g = 0$ and $g \geq 1$. Moreover, the result is more general than the corresponding one in [6] for the Kähler-Yang-Mills equations. We plan on exploring these equations further in future work.

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2. SEMI-STABLE AMPLE BUNDLES ON SURFACES

In this section we study stable Hartshorne-ample bundles on compact complex surfaces. We need to use two known facts.

- (1) Let E be a holomorphic vector bundle of rank r over a compact complex surface X . Let $L = \mathcal{O}_E(1)$ be the canonical bundle over $\mathbb{P}(E)$. At the level of classes it is well known that $\pi_*([c_1(L)^r]) = [c_1(E)]$ and $\pi_*([c_1^{r+1}(L)]) = [c_1^2(E) - c_2(E)]$ where π_* is the fibre-integral. As a consequence (see lemma 4.1.1 from [13] for instance) it follows that $c_1(E)$ and $c_1^2(E) - c_2(E)$ have positive representatives¹. In particular, X is projective by the Kodaira embedding theorem.

¹It follows from a theorem proven in [12, 8] that this equality holds even at the level of Chern-Weil forms for Griffiths-positive bundles.

- (2) A theorem of Fulton-Lazarsfeld [11] (building on the work of Bloch-Gieseker [2]) shows that for a Hartshorne-ample bundle over a projective surface, c_1, c_1^2, c_2 and $c_1^2 - c_2$ are all numerically positive, i.e., when integrated over subvarieties over appropriate dimensions one gets positive numbers. Actually this theorem holds true in greater generality (the only numerically positive classes are positive linear combinations of Schur polynomials of Chern classes). But for our purposes this statement is good enough.

Proof of theorem 1.1 :

Let ω be a metric in the class $c_1(L)$. The assumption of semi-stability with respect to the polarisation L and a theorem of Kobayashi (theorem 10.13 in [15]) shows that the rank- r bundle E is approximately Hermite-Einstein, i.e., for every given $1 > \epsilon > 0$, there exists a metric $h_{0,\epsilon}$ satisfying

$$(2.1) \quad \left\| \frac{\sqrt{-1}\Theta_{0,\epsilon} \wedge \omega}{\omega^2} - \lambda Id \right\| < \epsilon,$$

where $\Theta_{0,\epsilon}$ is the curvature of the Chern connection of $h_{0,\epsilon}$, and $\lambda = \frac{\int_X c_1(E)\omega}{r \int_X \omega^2}$ is a positive constant. For this metric the proof of the Kobayashi-Lübke inequality (theorem 5.7 in [15]) shows that

$$(2.2) \quad (r-1)c_1(E, h_{0,\epsilon})^2 - 2rc_2(E, h_{0,\epsilon}) \leq C\epsilon\omega^2,$$

where C is a constant depending only on λ, r .

We suppress the dependence on ϵ from now. It is not clear that the first Chern form satisfies $c_1(E, h_{0,\epsilon}) > 0$. If it did, then c_2 would be positive as well. We conformally change the metric $h = h_0 e^{-\phi}$ in the hope that for appropriately chosen ϕ this new metric satisfies the conditions of the theorem. We compute the new Chern-Weil forms :

$$(2.3) \quad \begin{aligned} \Theta_h &= \Theta_0 + \partial\bar{\partial}\phi Id \\ c_1(h) &= c_1(h_0) + r \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\phi \\ c_2(h) &= c_2(h_0) + (r-1) \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\phi \wedge c_1(h_0) + \frac{r(r-1)}{2} \left(\frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\phi \right)^2. \end{aligned}$$

Using equations 2.3 we compute the following linear combination of Chern forms (the second Segre form).

$$\begin{aligned}
c_1^2(h) - c_2(h) &= \left(c_1(h_0) + r \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \phi \right)^2 \\
&\quad - \left(c_2(h_0) + (r-1) \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \phi \wedge c_1(h_0) + \frac{r(r-1)}{2} \left(\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \phi \right)^2 \right) \\
&= c_1^2(h_0) - c_2(h_0) + (r+1) \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \phi \wedge c_1(h_0) + \frac{r(r+1)}{2} \left(\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \phi \right)^2 \\
&= c_1^2(h_0) - c_2(h_0) + \frac{r(r+1)}{2} \left(\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \phi + \frac{c_1(h_0)}{r} \right)^2 - \frac{(r+1)c_1(h_0)^2}{2r} \\
(2.4) \quad &= \frac{(r-1)c_1^2(h_0) - 2rc_2(h_0)}{2r} + \frac{r(r+1)}{2} \left(\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \phi + \frac{c_1(h_0)}{r} \right)^2.
\end{aligned}$$

As mentioned earlier, we know that $c_1^2 - c_2$ has a positive representative η . Therefore we may attempt to find a ϕ solving

$$(2.5) \quad \frac{r(r+1)}{2} \left(\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \phi + \frac{c_1(h_0)}{r} \right)^2 = \eta + \frac{2rc_2(h_0) - (r-1)c_1^2(h_0)}{2r}.$$

Thanks to inequality 2.2 we see that for sufficiently small ϵ the right hand side is positive. The class $[c_1(h_0)]$ is a Kähler class. Moreover, the integrals of both sides of the equation are equal. Indeed,

$$\begin{aligned}
&\int_X \frac{r(r+1)}{2} \left(\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \phi + \frac{c_1(h_0)}{r} \right)^2 = \int_X \frac{r(r+1)}{2} \left(\frac{c_1(h_0)}{r} \right)^2 \\
&\int_X \left(\eta + \frac{2rc_2(h_0) - (r-1)c_1^2(h_0)}{2r} \right) = \int_X \left(c_1^2 - c_2 + \frac{2rc_2 - (r-1)c_1^2}{2r} \right) \\
(2.6) \quad &= \int_X \frac{r(r+1)}{2} \left(\frac{c_1(h_0)}{r} \right)^2.
\end{aligned}$$

Therefore, by Yau's solution of the Calabi conjecture [22] we have a smooth function ϕ solving equation 2.5 such that $\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \phi + \frac{c_1(h_0)}{r} = \frac{c_1(h)}{r} > 0$. Computing the second Chern form using equations 2.5 and 2.3 we see that

$$c_2(h) = c_2(h_0) + \frac{r-1}{r+1} \left(\eta + c_2(h_0) - c_1^2(h_0) \right) = \frac{2rc_2(h_0) - (r-1)c_1^2(h_0) + (r-1)\eta}{r+1},$$

which is positive for small ϵ . Therefore we have successfully found a metric h on the bundle E satisfying

$$\left\| \frac{\sqrt{-1} \Theta_h \wedge \omega}{\omega^2} - (\lambda + \Delta \phi) Id \right\| < \epsilon, \quad c_1(h) > 0, \quad c_2(h) > 0, \quad \text{and} \quad c_1^2(h) - c_2(h) > 0.$$

□

In addition, we observe that

$$\begin{aligned}
 (r-1)c_1^2(h) - 2rc_2(h) &= (r-1) \left(c_1(h_0) + r \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \phi \right)^2 \\
 &\quad - 2r \left(c_2(h_0) + (r-1) \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \phi \wedge c_1(h_0) + \frac{r(r-1)}{2} \left(\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \phi \right)^2 \right) \\
 (2.7) \qquad \qquad \qquad &= (r-1)c_1^2(h_0) - 2rc_2(h_0) \leq C\epsilon\omega^2.
 \end{aligned}$$

It is tempting to hope that given the large number of conditions being satisfied, the curvature Θ_h of the metric h is Griffiths positive. But it is not clear and we suspect that it is unlikely.

3. CHERN FORMS OF A VORTEX BUNDLE

In order to prove theorem 1.2 we need to calculate the curvature of an $SU(2)$ -invariant metric H on E . We follow the calculations in [5, 17]. Consider the metric $H = h_1 \oplus f_2 \frac{8\pi}{\tau} \frac{dz \otimes d\bar{z}}{(1+|z|^2)^2}$ on $\pi_1^* L \oplus \pi_2^* \mathcal{O}(2)$ where h_1 is a metric on L over Σ , and f_2 is a function on Σ . For future use, define $h = \frac{h_1}{f_2}$ and $g_2 = f_2 \frac{8\pi}{\tau} \frac{dz \otimes d\bar{z}}{(1+|z|^2)^2}$. Notice that every holomorphic structure E on the smooth complex bundle $\pi_1^* L \oplus \pi_2^* \mathcal{O}(2)$ can be given by an element $\beta \in H^1(X, \pi_1^* L \otimes \pi_2^* \mathcal{O}(2)) = H^0(\Sigma, L)$. Every such β is of the form

$$\beta = \pi_1^* \phi \otimes \pi_2^* \zeta,$$

where $\phi \in H^0(\Sigma, L)$ and $\zeta = \frac{\sqrt{8\pi}}{\tau} \frac{dz \otimes d\bar{z}}{(1+|z|^2)^2}$. In an orthonormal frame the Chern connection on E associated to H and β is of the following form.

$$A = \begin{pmatrix} A_{h_1} & \beta \\ -\beta^* & A_{g_2} \end{pmatrix}.$$

The curvature is

$$(3.1) \qquad \Theta = \begin{pmatrix} \Theta_{h_1} - \beta \wedge \beta^* & \nabla^{(1,0)} \beta \\ -\nabla^{(0,1)} \beta^* & \Theta_{g_2} - \beta^* \wedge \beta \end{pmatrix}.$$

We note that [17]

$$\begin{aligned}
 \beta \wedge \beta^* &= \frac{\sqrt{-1}}{\tau} |\phi|_h^2 \omega_{FS} \\
 (3.2) \qquad \nabla^{(1,0)} \beta \wedge \nabla^{(0,1)} \beta^* &= -\frac{\sqrt{-1}}{\tau} \nabla^{(1,0)} \phi \wedge \nabla^{(0,1)} \phi^* \wedge \omega_{FS},
 \end{aligned}$$

where as before, $h = \frac{h_1}{f_2}$. Upon calculation (in normal coordinates) we see that

$$(3.3) \qquad \partial \bar{\partial} |\phi|_h^2 = -\Theta_h |\phi|_h^2 + \nabla^{(1,0)} \phi \wedge \nabla^{(0,1)} \phi^*.$$

Now we compute the terms in equation 1.5.

$$\begin{aligned}
 & \sqrt{-1}\Theta\Omega - \lambda\Omega^2 Id \\
 = & \begin{pmatrix} \sqrt{-1}\Theta_{h_1}\frac{4}{\tau}\omega_{FS} + \frac{|\phi|_h^2}{\tau}\omega_{FS}\omega_\Sigma - 2\lambda\frac{4}{\tau}\omega_\Sigma\omega_{FS} & 0 \\ 0 & (\sqrt{-1}\Theta_{f_2} + 2\omega_{FS})\Omega - \frac{|\phi|_h^2}{\tau}\omega_{FS}\omega_\Sigma - 2\lambda\frac{4}{\tau}\omega_\Sigma\omega_{FS} \end{pmatrix} \\
 (3.4) \quad & = \frac{4}{\tau}\omega_{FS} \begin{pmatrix} \sqrt{-1}(\Theta_h + \Theta_{f_2}) + (\frac{|\phi|_h^2}{4} - 2\lambda)\omega_\Sigma & 0 \\ 0 & \sqrt{-1}\Theta_{f_2} + (\frac{\tau}{2} - \frac{|\phi|_h^2}{4} - 2\lambda)\omega_\Sigma \end{pmatrix}
 \end{aligned}$$

$$\begin{aligned}
 \text{ch}_2(E, H) &= -\frac{1}{2(2\pi)^2} \text{Tr}(\Theta^2) \\
 &= -\frac{1}{2(2\pi)^2} \left((\Theta_{h_1} - \beta \wedge \beta^*)^2 - 2\nabla^{(1,0)}\beta \wedge \nabla^{(0,1)}\beta^* + (\Theta_{g_2} - \beta^* \wedge \beta)^2 \right) \\
 &= \frac{\sqrt{-1}}{\tau(2\pi)^2} \left(\Theta_{h_1}|\phi|_h^2\omega_{FS} - \nabla^{(1,0)}\phi \wedge \nabla^{(0,1)}\phi^* \wedge \omega_{FS} + 2\tau\Theta_{f_2}\omega_{FS} - \Theta_{f_2}|\phi|_h^2\omega_{FS} \right) \\
 (3.5) \quad &= \frac{\sqrt{-1}}{\tau(2\pi)^2} \omega_{FS} \left(\Theta_h|\phi|_h^2 - \nabla^{(1,0)}\phi \wedge \nabla^{(0,1)}\phi^* + 2\tau\Theta_{f_2} \right) = \frac{\sqrt{-1}}{\tau(2\pi)^2} \omega_{FS} \left(-\partial\bar{\partial}|\phi|_h^2 + 2\tau\Theta_{f_2} \right).
 \end{aligned}$$

Since η is $SU(2)$ -invariant, it is of the form $\eta = \frac{8}{\tau}f \wedge \omega_{FS}$ where f is the pull-back of a volume form from Σ . Therefore we calculate the second equation in the Calabi-Yang-Mills equations as follows.

$$(3.6) \quad \Omega^2 + \alpha \text{ch}_2(E, H) - \eta = \omega_{FS} \left(\frac{8}{\tau}\omega_\Sigma + \frac{\sqrt{-1}\alpha}{\tau(2\pi)^2}(-\partial\bar{\partial}|\phi|_h^2 + 2\tau\Theta_{f_2}) - \frac{8}{\tau}f \right).$$

For the record we note that (upon integration on both sides) $\lambda = \frac{\tau}{8} + \frac{c_1(L)\pi}{2\text{Vol}(\Sigma)}$. We now proceed to prove theorem 1.2.

Proof of part 1) of theorem 1.2 : Since $\int \Omega^2 = \int \eta$ it is clear that $\frac{8}{\tau}\omega_\Sigma - \frac{8}{\tau}f = \frac{\alpha}{\tau(2\pi)^2} \sqrt{-1}\partial\bar{\partial}u$ for some function u . Using equation 3.6 we see that the right-hand side is zero if and only if $|\phi|_h^2 + 2\tau \ln(f_2) = u$. Therefore we can easily choose (an infinite number of) f_2 satisfying the equation in the first part of theorem 1.2. \square

Proof of part 2) of theorem 1.2 : The Calabi-Yang-Mills equations are (using 3.4, 3.6)

$$\begin{aligned}
 \sqrt{-1}(\Theta_h + \Theta_{f_2}) + \left(\frac{|\phi|_h^2}{4} - 2\lambda\right)\omega_\Sigma &= 0 \\
 \sqrt{-1}\Theta_{f_2} + \left(\frac{\tau}{2} - \frac{|\phi|_h^2}{4} - 2\lambda\right)\omega_\Sigma &= 0 \\
 8\omega_\Sigma + \frac{\sqrt{-1}\alpha}{(2\pi)^2}(-\partial\bar{\partial}|\phi|_h^2 + 2\tau\Theta_{f_2}) - 8f &= 0.
 \end{aligned}
 \tag{3.7}$$

The case of $\alpha = 0$: When $\alpha = 0$, the third equation can be solved trivially for a unique, smooth ω_Σ . The other two can also be solved for a smooth solution because E is stable [17] and therefore Donaldson's existence theorem [9] applies. As for uniqueness, from the second equation it is clear that a unique h ensures a unique f_2 with zero-average. Substituting the second equation in the first, we get the following equation.

$$\sqrt{-1}\Theta_h + \left(\frac{|\phi|_h^2 - \tau}{2}\right)\omega_\Sigma = 0.
 \tag{3.8}$$

Suppose there are two solutions $\mathbf{h}_1 = \mathbf{h}_0 e^{-\psi_1}$ and $\mathbf{h}_2 = \mathbf{h}_0 e^{-\psi_2}$ where \mathbf{h}_0 is a metric on L , then upon subtraction we get

$$\sqrt{-1}\partial\bar{\partial}(\psi_2 - \psi_1) + \frac{|\phi|_{\mathbf{h}_2}^2}{2}(1 - e^{\psi_2 - \psi_1}) = 0.
 \tag{3.9}$$

At the global maximum of $\psi_2 - \psi_1$ we know that $\sqrt{-1}\partial\bar{\partial}(\psi_2 - \psi_1) \leq 0$. Therefore $1 \geq e^{\psi_2 - \psi_1}$. This means that $\psi_2 \leq \psi_1$ throughout. Interchanging the roles of ψ_1 and ψ_2 we see that $\psi_1 = \psi_2$.²

Before we proceed to prove openness along this ‘‘continuity path’’, we note that ω_Σ is always positive along this path. Indeed, if $\omega_\Sigma(p) = 0$ at some point p for the first such value of $\alpha > 0$, then the first two equations of 3.7 show that $\Theta_{h_1} = \Theta_{f_2} = 0$ at p . Choosing normal coordinates at p we see that the third equation implies

$$-\frac{\sqrt{-1}\alpha}{(2\pi)^2}\partial\phi(p) \wedge \bar{\partial}\bar{\phi}(p) = 8f(p) > 0.
 \tag{3.10}$$

This is a contradiction. Therefore, as long (measured in α) as a solution exists, $\omega_\Sigma > 0$. \square

Openness : Now we prove openness. Firstly, we claim that

$$|\phi|_h^2 - \tau \leq 0.
 \tag{3.11}$$

²Incidentally, equation 3.9 can be solved using the existence theorem of Kazdan-Warner [14]. This provides an alternative proof of existence.

Indeed, let the maximum of $|\phi|_h^2$ be attained at a point p . Then

$$0 \geq \sqrt{-1}\partial\bar{\partial}|\phi|_h^2(p) \geq -\Theta_h(p)|\phi|_h^2(p).$$

Using equation 3.8 we see that $|\phi|_h^2(p) \leq \tau$. Now consider the following Banach spaces/manifolds of functions on Σ equipped with the fixed background metric f . $\mathcal{B}_0^{k+2,a}$ is the subspace of $C^{k+2,a}$ Hölder functions with zero average, $\mathcal{B}_{>0}^{k+2,a}$ is the open subset of $\mathcal{B}_0^{k+2,a}$ with elements φ satisfying $\omega_\Sigma = f + \sqrt{-1}\partial\bar{\partial}\varphi > 0$, and $\mathcal{B}_{sub}^{k+2,a}$ is the submanifold of $\mathcal{B}^{k+2,a} \times \mathcal{B}_{>0}^{k+2,a}$ consisting of pairs of functions (ψ, φ) such that $\int_\Sigma |\phi|_{h_0}^2 e^{-\psi} (f + \sqrt{-1}\partial\bar{\partial}\varphi) = \text{Vol}(\Sigma)(2\tau - 8\lambda)$. Note that the tangent space $T\mathcal{B}_{sub}^{k+2,a}$ of $\mathcal{B}_{sub}^{k+2,a}$ at (ψ, φ) consists of functions $\dot{\psi}, \dot{\varphi}$ such that

$$(3.12) \quad \int \dot{\varphi} f = 0$$

$$\int \left(-|\phi|_{h_0}^2 e^{-\psi} \dot{\psi} (f + \sqrt{-1}\partial\bar{\partial}\varphi) + |\phi|_{h_0}^2 e^{-\psi} \sqrt{-1}\partial\bar{\partial}\dot{\varphi} \right) = 0.$$

Define the map $T : B_1 = \mathbb{R} \times \mathcal{B}_{sub}^{k+2,a} \times \mathcal{B}_0^{k+2,a} \rightarrow B_2 = \mathcal{B}_0^{k,a} \times \mathcal{B}_0^{k,a} \times \mathcal{B}_0^{k,a}$ as

$$(3.13) \quad T(\alpha, \psi, \varphi, \psi_2) = (T_1, T_2, T_3) \text{ where}$$

$$T_1 = \sqrt{-1}(\Theta_{h_0 e^{-\psi}} + \Theta_{e^{-\psi_2}}) + \left(\frac{|\phi|_{h_0}^2 e^{-\psi}}{4} - 2\lambda \right) (f + \sqrt{-1}\partial\bar{\partial}\varphi)$$

$$= \sqrt{-1}(\Theta_{h_0} + \partial\bar{\partial}\psi + \partial\bar{\partial}\psi_2) + \left(\frac{|\phi|_{h_0}^2 e^{-\psi}}{4} - 2\lambda \right) (f + \sqrt{-1}\partial\bar{\partial}\varphi)$$

$$T_2 = \sqrt{-1}\Theta_{e^{-\psi_2}} + \left(\frac{\tau}{2} - \frac{|\phi|_{h_0}^2 e^{-\psi}}{4} - 2\lambda \right) (f + \sqrt{-1}\partial\bar{\partial}\varphi)$$

$$= \sqrt{-1}\partial\bar{\partial}\psi_2 + \left(\frac{\tau}{2} - \frac{|\phi|_{h_0}^2 e^{-\psi}}{4} - 2\lambda \right) (f + \sqrt{-1}\partial\bar{\partial}\varphi)$$

$$T_3 = 8(f + \sqrt{-1}\partial\bar{\partial}\varphi) + \frac{\sqrt{-1}\alpha}{(2\pi)^2} (-\partial\bar{\partial}(|\phi|_{h_0}^2 e^{-\psi}) + 2\tau\Theta_{e^{-\psi_2}}) - 8f$$

$$= 8\sqrt{-1}\partial\bar{\partial}\varphi + \frac{\sqrt{-1}\alpha}{(2\pi)^2} (-\partial\bar{\partial}(|\phi|_{h_0}^2 e^{-\psi}) + 2\tau\partial\bar{\partial}\psi_2)$$

Clearly T is a smooth map. Assume that for a given $\alpha \geq 0$ there exists a ψ, ψ_2, φ such that $T(\alpha, \psi, \varphi, \psi_2) = 0$. We will show that the derivative DT evaluated at this point is a surjective Fredholm operator when acting on $(0, \dot{\psi}, \dot{\varphi}, \dot{\psi}_2) \in 0 \times T\mathcal{B}_{sub}^{k+2,a} \times \mathcal{B}_0^{k+2,a}$. By the implicit function theorem on Banach manifolds and the Fredholmness of DT , this will imply that T is

locally onto for an open neighbourhood of α . Indeed, the derivative DT is

$$\begin{aligned}
 DT(0, \dot{\psi}, \dot{\phi}, \dot{\psi}_2) &= (S_1, S_2, S_3) \text{ where} \\
 S_1 &= \sqrt{-1}\partial\bar{\partial}\dot{\psi} + \sqrt{-1}\partial\bar{\partial}\dot{\psi}_2 - \frac{|\phi|_h^2}{4}\dot{\psi}(f + \sqrt{-1}\partial\bar{\partial}\dot{\phi}) + \left(\frac{|\phi|_{h_0}^2 e^{-\psi}}{4} - 2\lambda\right)\sqrt{-1}\partial\bar{\partial}\dot{\phi} \\
 S_2 &= \sqrt{-1}\partial\bar{\partial}\dot{\psi}_2 + \frac{|\phi|_h^2}{4}\dot{\psi}(f + \sqrt{-1}\partial\bar{\partial}\dot{\phi}) + \left(\frac{\tau}{2} - \frac{|\phi|_{h_0}^2 e^{-\psi}}{4} - 2\lambda\right)\sqrt{-1}\partial\bar{\partial}\dot{\phi} \\
 (3.14) \quad S_3 &= 8\sqrt{-1}\partial\bar{\partial}\dot{\phi} + \frac{\sqrt{-1}\alpha}{(2\pi)^2}(\partial\bar{\partial}(|\phi|_{h_0}^2 e^{-\psi}\dot{\psi}) + 2\tau\partial\bar{\partial}\dot{\psi}_2).
 \end{aligned}$$

We have the following lemma.

Lemma 3.1. *The linearisation DT is an elliptic system.*

Proof. The principal symbol of DT is

$$(3.15) \quad \sqrt{-1}|\xi|^2 \begin{bmatrix} 1 & 1 & \left(\frac{|\phi|_{h_0}^2 e^{-\psi}}{4} - 2\lambda\right) \\ 0 & 1 & \left(\frac{\tau}{2} - \frac{|\phi|_{h_0}^2 e^{-\psi}}{4} - 2\lambda\right) \\ \frac{\alpha}{(2\pi)^2}|\phi|_{h_0}^2 e^{-\psi} & \frac{2\alpha\tau}{(2\pi)^2} & 8 \end{bmatrix}$$

It is clearly positive-definite if and only if its determinant is positive. The determinant is

$$8 + \frac{2\alpha\tau}{(2\pi)^2} \left[\frac{|\phi|_h^2}{4} + 2\lambda - \frac{\tau}{2} \right] + \frac{\alpha}{2(2\pi)^2} |\phi|_h^2 (\tau - |\phi|_h^2),$$

which is positive-definite because the first term is positive by assumption 1.4 and the second by inequality 3.11. Therefore the operator is strongly elliptic and is thus Fredholm. \square

By the Fredholm alternative, it is surjective if and only if its formal L^2 adjoint DT^* from B_2 to B_1 has a trivial kernel. Indeed the kernel of the formal adjoint consists of functions u, v, w of zero f -average such that

$$\begin{aligned}
 \sqrt{-1}\partial\bar{\partial}u - \frac{|\phi|_h^2}{4}u(f + \sqrt{-1}\partial\bar{\partial}\dot{\phi}) + \frac{|\phi|_h^2}{4}v(f + \sqrt{-1}\partial\bar{\partial}\dot{\phi}) + \frac{\sqrt{-1}\alpha}{(2\pi)^2}|\phi|_h^2\partial\bar{\partial}w &= 0 \\
 \sqrt{-1}\partial\bar{\partial}u + \sqrt{-1}\partial\bar{\partial}v + \frac{2\alpha\tau}{(2\pi)^2}\sqrt{-1}\partial\bar{\partial}w &= 0 \\
 (3.16) \quad \sqrt{-1}\partial\bar{\partial} \left[u \left(\frac{|\phi|_h^2}{4} - 2\lambda \right) \right] + \sqrt{-1}\partial\bar{\partial} \left[v \left(\frac{\tau}{2} - \frac{|\phi|_h^2}{4} - 2\lambda \right) \right] + 8\sqrt{-1}\partial\bar{\partial}w &= 0.
 \end{aligned}$$

Solving the second and third equations of 3.16 we get the following equations.

$$\begin{aligned} \sqrt{-1}\partial\bar{\partial}u - \frac{|\phi|_h^2 u}{4}(f + \sqrt{-1}\partial\bar{\partial}\phi) + \frac{|\phi|_h^2 v}{4}(f + \sqrt{-1}\partial\bar{\partial}\phi) + \frac{\sqrt{-1}\alpha}{(2\pi)^2}|\phi|_h^2\partial\bar{\partial}w &= 0 \\ u + v &= -\frac{2\alpha\tau}{(2\pi)^2}w \\ (3.17) \quad \left[u \left(\frac{|\phi|_h^2}{4} - 2\lambda \right) \right] + \left[v \left(\frac{\tau}{2} - \frac{|\phi|_h^2}{4} - 2\lambda \right) \right] + 8w &= \text{constant}. \end{aligned}$$

Define $q = u - v$. Therefore,

$$\begin{aligned} u &= \frac{1}{2} \left(q - \frac{2\alpha\tau}{(2\pi)^2}w \right) \\ (3.18) \quad v &= -\frac{1}{2} \left(q + \frac{2\alpha\tau}{(2\pi)^2}w \right). \end{aligned}$$

Writing equations 3.17 in terms of w and q we get the following equations.

$$\begin{aligned} \frac{1}{2} \sqrt{-1}\partial\bar{\partial}q - \frac{|\phi|_h^2}{4}q(f + \sqrt{-1}\partial\bar{\partial}\phi) + \frac{\alpha}{(2\pi)^2}(|\phi|_h^2 - \tau) \sqrt{-1}\partial\bar{\partial}w &= 0 \\ (3.19) \quad \frac{|\phi|_h^2 - \tau}{4}q + w \left(8 + \frac{2\pi c_1(L)\alpha\tau}{(2\pi)^2 \text{Vol}(\Sigma)} \right) &= \text{constant}. \end{aligned}$$

The following lemma completes the proof.

Lemma 3.2. *The system of equations 3.19 has a unique smooth solution $q = w = 0$.*

Proof. Substituting the second equation of 3.19 in the first, we get the following equation for q .

$$(3.20) \quad \frac{1}{2} \sqrt{-1}\partial\bar{\partial}q - \frac{|\phi|_h^2}{4}q\omega_\Sigma - \frac{\alpha(|\phi|_h^2 - \tau)}{(2\pi)^2 \left(8 + \frac{c_1(L)\alpha\tau}{2\pi \text{Vol}(\Sigma)} \right)} \sqrt{-1}\partial\bar{\partial} \left(\frac{|\phi|_h^2 - \tau}{4}q \right) = 0.$$

Multiplying equation 3.20 by q and integrating-by-parts we arrive at

$$\begin{aligned} -2 \left(8 + \frac{c_1(L)\alpha\tau}{2\pi \text{Vol}(\Sigma)} \right) \int \sqrt{-1}\partial q \wedge \bar{\partial}q - \left(8 + \frac{c_1(L)\alpha\tau}{2\pi \text{Vol}(\Sigma)} \right) \int |\phi|_h^2 q^2 \omega_\Sigma \\ (3.21) \quad + \frac{\alpha}{(2\pi)^2} \int \sqrt{-1}\partial \left((|\phi|_h^2 - \tau)q \right) \wedge \bar{\partial} \left((|\phi|_h^2 - \tau)q \right) = 0. \end{aligned}$$

Since $|a + b|^2 = \frac{\sqrt{-1}(a+b) \wedge (\bar{a} + \bar{b})}{\omega_\Sigma} \leq |a|^2 + |b|^2 + 2|a||b|$ we see that

$$\begin{aligned} \sqrt{-1}\partial \left((|\phi|_h^2 - \tau)q \right) \wedge \bar{\partial} \left((|\phi|_h^2 - \tau)q \right) \\ \leq (\sqrt{-1}q^2 |\phi|_h^2 \nabla^{1,0}\phi \wedge \nabla^{0,1}\phi^* + \sqrt{-1}(|\phi|_h^2 - \tau)^2 \partial q \wedge \bar{\partial}q) + 2|a||b|\omega_\Sigma \\ (3.22) \quad = (\sqrt{-1}q^2 |\phi|_h^2 (\partial\bar{\partial}|\phi|_h^2 + |\phi|_h^2 \Theta_h) + \sqrt{-1}(|\phi|_h^2 - \tau)^2 \partial q \wedge \bar{\partial}q) + 2|a||b|\omega_\Sigma. \end{aligned}$$

Using equations 3.7 and 3.8 we get

$$\begin{aligned}
 & \sqrt{-1} \partial \left((|\phi|_h^2 - \tau) q \right) \wedge \bar{\partial} \left((|\phi|_h^2 - \tau) q \right) \leq \sqrt{-1} (|\phi|_h^2 - \tau)^2 \partial q \wedge \bar{\partial} q \\
 & + |\phi|_h^2 q^2 \left(|\phi|_h^2 \frac{\tau - |\phi|_h^2}{2} \omega_\Sigma + \frac{8(2\pi)^2 (\omega_\Sigma - f)}{\alpha} + 2\tau \left(-\frac{\tau}{4} + \frac{c_1(L)\pi}{\text{Vol}(\Sigma)} + \frac{|\phi|_h^2}{4} \right) \omega_\Sigma \right) + 2|a||b|\omega_\Sigma \\
 & \leq \sqrt{-1} (|\phi|_h^2 - \tau)^2 \partial q \wedge \bar{\partial} q + 2|a||b|\omega_\Sigma \\
 (3.23) \quad & + |\phi|_h^2 q^2 \left(|\phi|_h^2 \frac{\tau - |\phi|_h^2}{2} \omega_\Sigma + \frac{8(2\pi)^2 \omega_\Sigma}{\alpha} + 2\tau \left(-\frac{\tau}{4} + \frac{c_1(L)\pi}{\text{Vol}(\Sigma)} + \frac{|\phi|_h^2}{4} \right) \omega_\Sigma \right).
 \end{aligned}$$

Note that equality holds in the last inequality of 3.23 if and only if $q = 0$. Substituting 3.23 in 3.21 and simplifying we get

$$\begin{aligned}
 0 \leq & \left(-2 \left(8 + \frac{c_1(L)\alpha\tau}{2\pi\text{Vol}(\Sigma)} \right) + \frac{\alpha(|\phi|_h^2 - \tau)^2}{4\pi^2} \right) \int \sqrt{-1} \partial q \wedge \bar{\partial} q + 2 \frac{\alpha}{(2\pi)^2} \int |a||b|\omega_\Sigma \\
 (3.24) \quad & - \frac{\alpha}{8\pi^2} \int |\phi|_h^2 q^2 (\tau - |\phi|_h^2)^2 \omega_\Sigma
 \end{aligned}$$

Using the Cauchy-Schwartz inequality on the mixed term we get

$$\begin{aligned}
 0 \leq & \left(-2 \left(8 + \frac{c_1(L)\alpha\tau}{2\pi\text{Vol}(\Sigma)} \right) + \frac{\alpha(|\phi|_h^2 - \tau)^2}{4\pi^2} \right) \int \sqrt{-1} \partial q \wedge \bar{\partial} q - \frac{\alpha}{8\pi^2} \int |\phi|_h^2 q^2 (\tau - |\phi|_h^2)^2 \omega_\Sigma \\
 (3.25) \quad & + \frac{\alpha}{(2\pi)^2} \int \frac{\sqrt{-1} a \wedge \bar{a}}{g} + \frac{\alpha}{(2\pi)^2} \int g \sqrt{-1} b \wedge \bar{b},
 \end{aligned}$$

where we choose the function g to be (again using assumption 1.4)

$$(3.26) \quad g = \frac{\alpha(\tau - |\phi|_h^2)^2}{8\pi^2} \frac{1}{-\frac{\alpha(\tau - |\phi|_h^2)^2}{8\pi^2} + 8 + \frac{c_1(L)\tau\alpha}{2\pi\text{Vol}(\Sigma)}}.$$

Upon substitution of this g into 3.25 and simplifying we get $0 \leq 0$ and therefore $q = 0$. This means that $w = 0$ and hence $u = v = 0$. \square

This ends the proof of openness of the set of α (satisfying the assumptions of the theorem) for which the equation has a solution. \square

Regularity : In order to complete the proof we need to address the issue of regularity of solutions, i.e., if there is a $C^{2,a}$ solution of 3.7 (where ω_Σ is in $C^{0,a}$), is it smooth? (Since this is a semilinear system of equations we need to be a little careful.) Using the second and third equations of 3.7 we can solve for ω_Σ as follows.

$$(3.27) \quad \omega_\Sigma \left[4 + \frac{\tau\alpha}{(2\pi)^2} \left(2\lambda - \frac{\tau}{2} + \frac{|\phi|_h^2}{4} \right) \right] = 4f + \frac{\sqrt{-1}\alpha}{2(2\pi)^2} \partial \bar{\partial} |\phi|_h^2.$$

Note that

$$(3.28) \quad 4 + \frac{\tau\alpha}{(2\pi)^2} \left(2\lambda - \frac{\tau}{2}\right) > 0$$

by assumption 1.4. Substituting equation 3.27 in 3.8 and using 3.3 we get

$$(3.29) \quad \sqrt{-1}\Theta_h + \frac{|\phi|_h^2 - \tau 4f + \frac{\sqrt{-1}\alpha}{2(2\pi)^2}(-\Theta_h|\phi|_h^2 + \nabla^{(1,0)}\phi \wedge \nabla^{(0,1)}\phi^*)}{2 \left(4 + \frac{\tau\alpha}{(2\pi)^2} \left(2\lambda - \frac{\tau}{2} + \frac{|\phi|_h^2}{4}\right)\right)} = 0.$$

Writing $h = h_0 e^{-\psi}$ we get an equation of the form

$$(3.30) \quad F_1(e^{-\psi}, x)\Delta\psi = F_2(e^{-\psi}, \nabla\psi, x)$$

where $F_1 > 0$. Therefore, if ψ is in $C^{2,a}$ then since the right hand side is in $C^{1,a}$ by elliptic regularity ψ is in $C^{3,a}$. By bootstrapping ψ is smooth. This clearly implies that ω_Σ and f_2 are also smooth. \square

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